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### Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems

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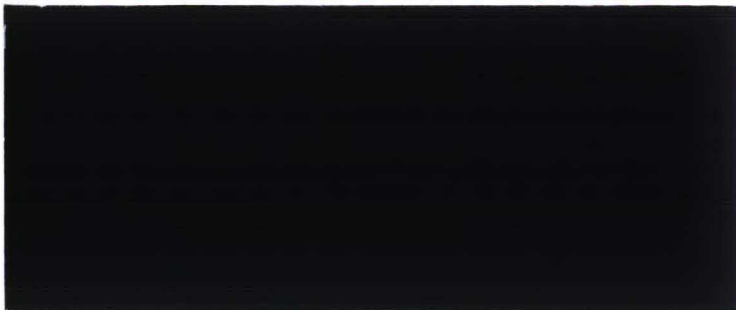
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REGULARITY AND SINGULARITY IN LINEAR-  
QUADRATIC CONTROL SUBJECT TO IMPLICIT  
CONTINUOUS-TIME SYSTEMS

Ton Geerts

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*Optimal Control*

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REGULARITY AND SINGULARITY IN LINEAR-QUADRATIC CONTROL  
SUBJECT TO IMPLICIT CONTINUOUS-TIME SYSTEMS

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ABSTRACT

A linear-quadratic (LQ) control problem subject to a standard continuous-time system is called regular if the input weighting matrix is invertible and singular if this is not the case. Consequently, optimal inputs for regular LQ problems are ordinary functions (state feedbacks), whereas optimal controls for singular problems are in general distributions, e.g. impulses. We will show that regularity and singularity in LQ problems subject to *general (implicit)* systems depends not so much on the input weighting matrix, as on the property that the integrand of the cost criterion is a function only if inputs and state trajectories are, as is the case for LQ problems subject to *standard* systems. In particular, we will provide a simple criterion for distinguishing between regularity and singularity in LQ problems subject to *general* systems. Our criterion is expressed in the system coefficients only and reduces to the classical one if the underlying systems are standard.

KEYWORDS

Linear-quadratic optimal control problems, implicit systems, distributions, strongly controllable subspace, left invertibility.

# 1. Introduction.

Consider the following standard Linear-Quadratic Control Problem (see text-books like e.g. [1] - [4]).

(LQCP)<sub>1</sub>:

For all  $x_0$ , determine

$$J^+(x_0) := \inf_1 \int_0^\infty [x^2 + u^2] dt \mid u \in L_{2,loc}(\mathbb{R}^+), \lim_{t \rightarrow \infty} x(t) = 0,$$

subject to  $\dot{x}(t) = u(t)$ ,  $t \geq 0$ ,  $x(0) = x_0$ ,

with  $L_{2,loc}(\mathbb{R}^+)$  denoting the class of locally square-integrable functions on  $\mathbb{R}^+ := [0, \infty)$ . The problem is *regular* in the sense of Hilbert [5, p. 29] since the "weighting" scalar of the input  $u$  in the cost criterion  $\int_0^\infty [x^2 + u^2] dt$ , 1, is invertible as a result of which optimal controls are state feedbacks and hence ordinary functions [1] - [4], [6, Corollary 3.4]. In particular, we have here that  $\bar{u}$ , the optimal input, can be written as

$$\bar{u} = -\bar{x},$$

with  $\bar{x}$  denoting the optimal state trajectory  $e^{-t}x_0$ ,  $t \geq 0$ .

Next, consider

(LQCP)<sub>2</sub>:

For all  $x_0$ , determine

$$J^+(x_0) := \inf_2 \int_0^\infty x^2 dt \mid u \in L_{2,loc}(\mathbb{R}^+), \lim_{t \rightarrow \infty} x(t) = 0,$$

subject to  $\dot{x}(t) = u(t)$ ,  $t \geq 0$ ,  $x(0) = x_0$ .

Since the weighting scalar of the control in the cost criterion is obviously singular (not invertible), this LQCP is called a singular problem. A typical aspect of singular LQCPs is the fact that optimal inputs as well as optimal state trajectories may *not* exist within the class of ordinary (measurable) functions. For instance, it is easily seen that for every  $\epsilon > 0$  and every  $x_0$  the control  $u = -x_0^2(2\epsilon)^{-1}\exp(-x_0^2(2\epsilon)^{-1}t)x_0$  yields  $x = \exp(-x_0^2(2\epsilon)^{-1}t)x_0$  and  $\int_0^\infty x^2 dt = \epsilon$ , but an optimal control exists within  $L_{2,loc}(\mathbb{R}^+)$  only if  $x_0 = 0$ . In fact, if  $x_0 \neq 0$ , then it is suggested [7] that the *impulsive* control  $u = -\delta(t)x_0$ , with  $\delta(t)$  denoting Diracs delta function, is optimal for the latter LQCP since this input yields  $x(t) = 0$  for  $t > 0$ .



In [8] - [9] it is demonstrated that singular LQCPs can be solved in full detail by allowing certain generalized functions (distributions [10]) as inputs and state trajectories. What is more, since any LQCP can be redefined in terms of distributions, it is easily seen that regularity of classical LQCP is equivalent to the property that the integrand in the associated cost criterion is a function *only if* the involved inputs and state trajectories are functions as well (see Section 2). Indeed, in  $(LQCP)_1$ , a regular problem, the integrand  $[x^2 + u^2]$  is a function only if  $u$  and  $x$  are, but in  $(LQCP)_2$ , a singular problem, the integrand  $x^2$  may be a function, whereas  $u$  is not (see the above).

Now, let us observe an LQCP subject to an *implicit* system in the formulation of [11] - [12] (for details, see Section 2):

For all  $\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ , determine

$$\inf \left\{ \int_0^\infty [x_1^2 + 2x_2^2] dt \mid u \in L_{2,loc}(\mathbb{R}^+), \lim_{t \rightarrow \infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0 \right\},$$

subject to

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad t \geq 0, \quad \begin{bmatrix} x_1(0^-) \\ x_2(0^-) \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

The integrand of the cost criterion does not contain a control term. Yet the LQCP is regular in the sense that optimal controls and state trajectories are ordinary functions (see [11] - [12]). Indeed, since  $u = -x_2$ , we have that the integrand of the cost criterion equals  $[x_1^2 + x_2^2 + u^2]$  and hence this integrand is a function only if  $u$ ,  $x_1$  and  $x_2$  are.

Also, consider the same implicit system with the criterion:

For all  $\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ , determine

$$\inf \left\{ \int_0^\infty u^2 dt \mid u \in L_{2,loc}(\mathbb{R}^+), \lim_{t \rightarrow \infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 0 \right\}.$$

The weighting scalar of the control is invertible. Yet the problem is singular in the sense that optimal controls and/or state trajectories need *not* be ordinary functions: It is stated in [13], [11] that the solution of the given system with  $u = 0$  is  $x_2 = 0$ ,  $x_1 = -x_{02}\delta(t)$  and hence the optimal state trajectory is *impulsive* if  $x_{02} \neq 0$ .

These two examples of LQCPs subject to implicit systems demonstrate that regularity and singularity of LQCPs subject to *arbitrary* systems may *not* depend solely on the invertibility of the control's weighting matrix in the cost criterion. After the preliminaries in Section 2, we will prove in Section 3 of the present paper that regularity of any LQCP is *equivalent* to the system property that the associated cost criterion integrand is a function (if and) only if inputs and state trajectories are, as is the case if the LQCP is classical, see Section 2. This property will be expressed in the system coefficients only. In particular, this condition reduces to the classical one (invertibility of the control's weighting matrix in the cost criterion) if the underlying system is standard, i.e., not implicit.

## 2. Preliminaries.

The first extensive treatment of singular LQCPs subject to standard systems is given in [8]. The proposed class of allowed distributions  $C_{imp}$  in [8] turns out to be large enough to be representative for the system's behaviour in LQCPs subject to standard systems ([9], [14], [6], [15]). Since, moreover,  $C_{imp}$  has many nice properties, we will adopt the class  $C_{imp}$  for defining LQCPs subject to arbitrary (possibly implicit) systems - compare the choice of distributions in [11].

The class  $C_{imp}$  is investigated in detail in [16], see also [8], [15]; we will recall a few main points. A distribution  $u \in C_{imp}$  is called *impulsive-smooth* and it can be decomposed (uniquely!) in an *impulsive* part  $u_1$  and a *smooth* part  $u_2$ . A distribution is called *impulsive* if it is a linear combination of the Dirac delta distribution  $\delta$  and its distributional derivatives  $\delta^{(i)}$ ,  $i \geq 1$  (for details on distributions, see Schwartz [10]). A *smooth* distribution is a function which is smooth on  $\mathbb{R}^+ := [0, \infty)$  [8, Definition 3.1] and zero elsewhere. The class  $C_{imp}$  is a commutative algebra over  $\mathbb{R}$  with convolution \* of distributions as multiplication (unit element  $\delta$ ) and hence it is closed under differentiation (= convolution with  $\delta^{(1)}$ ) and closed under integration (= convolution with  $H$ , the Heaviside "unit step" distribution). It holds that  $\delta^{(i)} = \delta^{(i-1)} * \delta^{(1)}$  ( $i \geq 1$ ) with  $\delta^{(0)} = \delta$ . By defining  $\delta^{(-1)} := H$ ,  $\delta^{(-j)} = \delta^{-(j-1)} * \delta^{(-1)}$  ( $j \geq 1$ ), we establish that  $\delta^{(i+j)} = \delta^{(i)} * \delta^{(j)}$  ( $i, j \in \mathbb{Z}$ ) and thus the *inverse* of  $\delta^{(i)}$ ,  $(\delta^{(i)})^{-1}$ , equals  $\delta^{(-i)}$  ( $i \in \mathbb{Z}$ ),  $(\delta)^{-1} = \delta$ ,  $\delta^{(-j)}$  is smooth and  $\delta^{(-j)}(t) = t^{j-1}/(j-1)!$  on  $\mathbb{R}^+$  and 0 elsewhere for  $j \geq 1$ . If  $C_{p-imp}$ ,  $C_{sm} \in C_{imp}$  denote the subalgebras of purely impulsive and smooth distributions, respectively, and  $u \in C_{sm}$ , then  $u(0^+) := \lim_{t \downarrow 0} u(t)$  and then the *distributional* derivative of  $u$ ,  $\delta^{(1)} * u$ , equals  $\dot{u} + u(0^+)\delta$ , where  $\dot{u}$  denotes the *ordinary* derivative of  $u$  on  $\mathbb{R}^+$ .



Now the *nonnegative definite* LQCP (with stability [17]) subject to a *standard* system can be stated as follows [8] - [9].

Given the system  $\Sigma$ :

$$\delta^{(1)} * x = Ax + Bu + x_0 \delta, \quad (2.1a)$$

$$y = Cx + Du, \quad (2.1b)$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{r \times n}$ ,  $D \in \mathbb{R}^{r \times m}$ ,  $x_0 \in \mathbb{R}^n$ ,  $u \in \mathcal{C}_{\text{imp}}^m$

(the  $m$ -vector version of  $\mathcal{C}_{\text{imp}}$ ). It can be shown without difficulty that  $(I_n \delta^{(1)} - A\delta)$  is invertible within  $\mathcal{C}_{\text{imp}}^{n \times n}$  with inverse corresponding to  $\exp(At)$  on  $\mathbb{R}^+$  [8, p. 375] (for example, the inverse of  $(\delta^{(1)} - 2\delta)$  equals  $u = \exp(2t)$  on  $\mathbb{R}^+$ , since  $(\delta^{(1)} - 2\delta) * u = \dot{u} + \delta u(0^+) - 2u = \delta$ ). Hence for every  $x_0 \in \mathbb{R}^n$  and every  $u \in \mathcal{C}_{\text{imp}}^m$  the equation (2.1a) has exactly one solution  $x = (I_n \delta^{(1)} - A\delta)^{-1} * (Bu + x_0 \delta) \in \mathcal{C}_{\text{imp}}^n$ . If  $u \in \mathcal{C}_{\text{sm}}^m$ , then  $x \in \mathcal{C}_{\text{sm}}^n$  and  $x = \exp(At)x_0 + \int_0^t \exp(A(t-r))Bu(r)dr$  on  $\mathbb{R}^+$ , i.e.,  $x$  equals the ordinary solution of  $\dot{x} = Ax + Bu$ ,  $x(0) = x_0$ , on  $\mathbb{R}^+$ , and we establish that the distributional framework (2.1) covers the usual (functional) one (e.g. [1] - [4], [8] - [9], [17]) if  $u \in \mathcal{C}_{\text{imp}}^m$  is a function.

Now, determine for every  $x_0$ ,

$$J^+(x_0) := \inf \left\{ \int_0^\infty Y' y dt \mid u \in \mathcal{C}_{\text{sm}}^m, \lim_{t \rightarrow \infty} x(t) = 0 \right\}. \quad (2.2)$$

The optimal cost is finite (i.e.,  $\forall x_0 \in \mathbb{R}^n: J^+(x_0) < \infty$ ) if and only if  $(A, B)$  is stabilizable (e.g. [9]). Assume this to be the case. If

$$J_D^+(x_0) := \inf \left\{ \int_0^\infty Y' y dt \mid u \in \mathcal{C}_{\text{imp}}^m, \lim_{t \rightarrow \infty} x(t) = 0 \right\}, \quad (2.3)$$

then

$$\forall x_0 \in \mathbb{R}^n: J^+(x_0) = J_D^+(x_0) \quad (2.4)$$

[15, Proposition 2.24], and optimal controls (if any) are in general distributions  $\in \mathcal{C}_{\text{imp}}^m$  if  $\ker(D) \neq 0$ , i.e., if the LQCP is singular [8], [9], [15, Section 3.2]. If  $\ker(D) = 0$ , however, then optimal inputs turn out to be ordinary functions. Our first result tells even more: If  $\ker(D) = 0$ , then the output  $y$  cannot be a function if the input  $u$  is not.

Proposition 2.1.

The LQCP (2.1) - (2.3) is regular if and only if, for every  $x_0$ ,  
 $y \in C_{sm}^r \Leftrightarrow u \in C_{sm}^m, x = (I_n \delta^{(1)} - A\delta)^{-1} * (Bu + x_0\delta) \in C_{sm}^n$ .

Proof.  $\Rightarrow$  Assume that  $\ker(D) = 0$  and  $y = Cx + Du$  is smooth. If  $v := u + (D'D)^{-1}D'Cx$ , then  $y = C_0x + Dv$ ,  $C_0 := (I - D(D'D)^{-1}D')C$ , and hence  $D'y = D'Dv$ , smooth. Thus,  $v$  smooth and since  $\delta^{(1)} * x = A_0x + Bv + x_0\delta$ ,  $A_0 := A - B(D'D)^{-1}D'C$ , it follows that  $x$  and  $u$  are smooth.  $\Leftarrow$  Assume that  $Dv = 0$  for some  $v \in \mathbb{R}^m$ . Then  $x = 0$  is the solution of (2.1a) with  $u := v\delta$  and  $x_0 := -Bv$ , and the output  $y$  equals 0. Hence  $u$  must be smooth, i.e.,  $v = 0$ .

Proposition 2.1 shows that, within a distributional setup, regularity of (2.1) - (2.3) is equivalent to the property that the output  $y$  is a function only if  $u$  and  $x$  are. In Section 3 we shall see that this property is equivalent to regularity of a nonnegative definite LQCP subject to any (possibly implicit) system. We will define such a problem (with stability) as follows.

Definition 2.2.

Given the system  $\Sigma$ :

$$E\delta^{(1)} * x = Ax + Bu + Ex_0\delta, \quad (2.5a)$$

$$y = Cx + Du, \quad (2.5b)$$

with  $E, A \in \mathbb{R}^{l \times n}$ ,  $B \in \mathbb{R}^{l \times m}$ ,  $C \in \mathbb{R}^{r \times n}$ ,  $D \in \mathbb{R}^{r \times m}$ , together with, for every  $(x_0, u) \in \mathbb{R}^n \times C_{imp}^m$ , the solution set  $S(x_0, u) :=$

$$\{x \in C_{imp}^n \mid [E\delta^{(1)} - A\delta] * x = Bu + Ex_0\delta\}. \quad (2.5c)$$

Then, determine, for every  $x_0$ ,  $J^+(x_0) :=$

$$\inf \left\{ \int_0^\infty y'y dt \mid \begin{bmatrix} u \\ x \end{bmatrix} \in C_{sm}^{m+n}, x \in S(x_0, u), \lim_{t \rightarrow \infty} x(t) = 0 \right\}. \quad (2.6)$$

# Discussion.

No assumptions are made on the system coefficients  $E$ ,  $A$ ,  $B$ ,  $C$  and  $D$ . In particular,  $E$  and  $A$  are allowed to be nonsquare. If  $E = I$ , then (2.5) - (2.6) reduces to (2.1) - (2.2), see the above or [8, Section 3]. More generally, if  $E$  is singular, but  $\det(sE - A) \neq 0$ , then  $S(x_0, u)$  contains exactly one element  $x = x(x_0, u)$  for every pair  $(x_0, u) \in \mathbb{R}^n \times C_{\text{imp}}^m$  - yet,  $x$  may have an impulsive part even if  $u$  is smooth: The distributional version of the implicit system in Section 1 is

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \delta^{(1)} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \delta$$

(see e.g. [11]) and  $u = 0$  yields  $x_2 = 0$ ,  $x_1 = -x_{02}\delta$ , impulsive. In [18, Proposition 3.5] it is shown why  $x$  is not automatically smooth if  $u$  is, as is the case with  $E = I$ .

## Proposition 2.3.

Assume that  $\det(sE - A) \neq 0$ . Let  $x_0 \in \mathbb{R}^n$ ,  $u \in C_{\text{sm}}^m$ . Then  $x(x_0, u) \in S(x_0, u) \cap C_{\text{sm}}^n$  if and only if  $E(x(x_0, u)(0^+)) = Ex_0$ .

In [11] - [12] it is assumed that  $\det(sE - A) \neq 0$ . Since in [11],  $C'C = I_n$ ,  $D'D = I_m$  and  $C'D = 0$ , it is clear that the LQCP in [11] and Definition 2.1 are identical. Note that the cost criterion's integrand in [11] is a function only if  $u$  and  $x$  are; the problem is regular in the sense that optimal controls and optimal state trajectories are functions. If  $D$  is merely of full column rank, then (2.5) - (2.6) reduces to the problem formulation in [12], because of Proposition 2.3. However, we will show that there are problems of the form in [12] that are *singular* in the sense that optimal inputs and state trajectories can be distributions, whereas there are also problems (2.5) - (2.6) with  $\det(sE - A) \neq 0$  and  $D$  *not* injective - yet the optimal inputs and associated state trajectories are ordinary functions.

In [11] as well as in [12] it is noted that the optimal state trajectories may be discontinuous in 0 in the sense that  $x(0^+)$  may be unequal to  $x_0$ . In fact, not so much  $x$  as  $Ex$  plays the role of "state" whose trajectory is optimized - and  $Ex(0^+) = Ex_0$  if input and state trajectory are functions, according to Proposition 2.3. If  $\det(sE - A) = 0$ , however,  $Ex(0^+)$  may be equal to  $Ex_0$  even if  $u$  and/or  $x \in S(x_0, u)$  are not smooth (see [18, Example 2.7]).

Our distributional formulation for implicit systems on  $\mathbb{R}^+$  (2.7a) is in line not only with earlier papers on the subject like [11], [19] - [20], but also with papers like [13] that are based on the Laplace transformation approach of Doetsch [21, § 22]. Moreover, we can keep our treatment fully *algebraic* because of our choice for  $C_{\text{imp}}$  as allowed class of distributions. Also, it can be easily shown that if  $x_0$  is consistent, i.e., if the ordinary differential-algebraic equation (DAE)  $E\dot{x} = Ax + Bu$  in the sense of Gantmacher [22] has for a certain function  $u$  a functional solution  $x$  with  $x(0^+) = x_0$ , then the distribution  $x \in S(x_0, u)$ . In other words, our approach covers the usual interpretation of singular DAEs as well (for an extensive investigation of (2.1a), see the recent [23], also [18]). Note, that the set  $S(x_0, u)$  in (2.5c) may be empty or even contain infinitely many solutions for certain pairs  $(x_0, u) \in \mathbb{R}^n \times C_{\text{imp}}^m$  since the pencil  $sE - A$  may even be nonsquare [22].

We close this Section with the concept of *strongly controllable* subspace [24, Definition 3.2], [25, Definition 3.1].

Definition 2.4.

A point  $x_0 \in \mathbb{R}^n$  is called *strongly controllable* if there exists an input  $u \in C_{p\text{-imp}}^m$  and a state trajectory  $x \in S(x_0, u) \cap C_{p\text{-imp}}^n$  such that  $y = 0$ . The space of these points is denoted by  $\mathcal{W}(\Sigma)$ .

If  $E = I$ , then our  $\mathcal{W}(\Sigma)$  and the one in [8] coincide.



### 3. Regularity and singularity.

Backed by Proposition 2.1, we make the following definition for regularity of the LQCP (2.5) - (2.6).

Definition 3.1.

The LQCP (2.5) - (2.6) is regular if, for every  $x_0 \in \mathbb{R}^n$ ,

$$y \in \mathbb{C}_{sm}^r \Leftrightarrow u \in \mathbb{C}_{sm}^m, x \in S(x_0, u) \cap \mathbb{C}_{sm}^n \quad (3.1)$$

and singular if this is not the case.

The first three examples in Section 1 are regular (in accordance with Proposition 2.1 and [11] - [12]), whereas the fourth example is singular, although the weighting matrix of the control in the associated cost criterion is invertible. In the proof of our key result Theorem 3.2 we will need the Main Lemma from [23], see also [18], [24]. For the reader's benefit, the simple proof of the Lemma is included.

Main Lemma.

Let  $x_0 \in \mathbb{R}^n$ ,  $u = u_1 + u_2$ ,  $u_1 \in \mathbb{C}_{p\text{-imp}}^m$ ,  $u_2 \in \mathbb{C}_{sm}^m$ , and  $x \in S(x_0, u)$ ,  $x = x_1 + x_2$ ,  $x_1 \in \mathbb{C}_{p\text{-imp}}^n$ ,  $x_2 \in \mathbb{C}_{sm}^n$ . Then

$$E\delta^{(1)} * x_1 + E(x_2(0^+))\delta = Ax_1 + Bu_1 + Ex_0\delta, \quad (3.2a)$$

$$E\delta^{(1)} * x_2 = Ax_2 + Bu_2 + E(x_2(0^+))\delta. \quad (3.2b)$$

Proof. Since  $E\delta^{(1)} * x_1 + E(x_2(0^+))\delta + \{E[\delta^{(1)} * x_2 - x_2(0^+)\delta]\} = Ax_1 + Bu_1 + Ex_0\delta + \{Ax_2 + Bu_2\}$  and  $\delta^{(1)} * x_2 - x_2(0^+)\delta = \dot{x}_2$ , the smooth derivative of  $x_2$  on  $\mathbb{R}^+$ , the claims are clear.

Theorem 3.2.

The LQCP (2.5) - (2.6) is regular if and only if

$$\ker \begin{pmatrix} E & 0 \\ C & D \end{pmatrix} \cap [A \ B]^{-1} \text{im}(E) = 0. \quad (3.3)$$



Proof. Let the LQCP be regular. If  $\bar{x} \in \mathbb{R}^n$ ,  $\bar{u} \in \mathbb{R}^m$  are such that  $E\bar{x} = 0$ ,  $C\bar{x} + D\bar{u} = 0$ ,  $A\bar{x} + B\bar{u} = Ew$  for a certain  $w \in \mathbb{R}^n$ , then  $\bar{x}\delta \in S(x_0, u)$  with  $x_0 := -w$ ,  $u := \bar{u}\delta$ , and the associated output  $y$  equals  $C\bar{x} + D\bar{u} = 0$ , regular. Hence  $\bar{x} = 0$ ,  $\bar{u} = 0$ . Conversely, let (3.3) be valid. It is proven in [25, Theorem 3.9] that  $\mathfrak{W}(\Sigma)$  (Definition 2.4) is the smallest subspace  $\mathcal{L}$  for which

$$E^{-1}[A \ B] \{(\mathcal{L} \oplus \mathbb{R}^m) \cap \ker([C \ D])\} \subset \mathcal{L}. \quad (3.4)$$

Hence  $\mathfrak{W}(\Sigma) \subset \ker(E)$ , since  $\ker(E)$  satisfies (3.4). On the other hand, trivially,  $\ker(E) \subset \mathfrak{W}(\Sigma)$ , and thus  $\mathfrak{W}(\Sigma) = \ker(E)$ . Now, let  $x_0 \in \mathbb{R}^n$ ,  $u \in \mathbb{C}_{\text{imp}}^m$ ,  $x \in S(x_0, u)$  be such that  $y$  is smooth. If  $u = u_1 + u_2$ ,  $x = x_1 + x_2$ ,  $u_1 \in \mathbb{C}_{\text{p-imp}}^m$ ,  $u_2 \in \mathbb{C}_{\text{sm}}^m$ ,  $x_1 \in \mathbb{C}_{\text{p-imp}}^n$ ,  $x_2 \in \mathbb{C}_{\text{sm}}^n$ , then we must show that  $x_1 = 0$  and  $u_1 = 0$ . By (3.2a),

$$E\delta^{(1)} * x_1 = Ax_1 + Bu_1 + E(x_0 - x_2(0^+))\delta,$$

$$y_1 := Cx_1 + Du_1 = 0,$$

and hence  $x_0 - x_2(0^+) \in \mathfrak{W}(\Sigma)$  (Definition 2.4). It follows that  $E(x_2(0^+)) = Ex_0$  and hence  $\begin{bmatrix} A\delta - E\delta^{(1)} & B\delta \\ C\delta & D\delta \end{bmatrix} * \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} = 0$ . By [25, Proposition 2.3, Corollary 2.4] (see also Remark 3.4), we establish that  $\begin{bmatrix} x_1 \\ u_1 \end{bmatrix} = 0$  if Rosenbrock's system matrix  $P_\Sigma(s) := \begin{bmatrix} A - sE & B \\ C & D \end{bmatrix}$  [26] is left invertible as a rational matrix. Without loss of generality, assume that the system  $\Sigma$  (2.5a) - (2.5b) is in the form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \delta^{(1)} * \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{01} \\ \bar{x}_{02} \end{bmatrix} \delta,$$

$$y = [C_1 \ C_2] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + Du.$$

Then the condition (3.3) is equivalent to left-invertibility of  $\begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}$ , as a result of which  $P_\Sigma(s)$  is indeed left invertible by Schur's lemma, and the proof is complete.

Remark 3.3.

If  $E = I$ , then (3.3) reduces to:  $\ker(D) = 0$ . Hence Theorem 3.2 covers the usual notion of regularity in (2.1) - (2.2).

Remark 3.4.

If  $C_f \in C_{\text{imp}}$  denotes the subalgebra of *fractional impulses*:

$$C_f := \{u \in C_{\text{imp}} \mid u = u_1 * u_2^{-1}, u_1, u_2 \in C_{p\text{-imp}}, u_2 \neq 0\}$$

( $u_2^{-1}$  denoting the inverse (w.r.t. convolution) of  $u_2$ ), then  $C_f$  is isomorphic to the commutative field of rational functions  $\mathbb{R}(s)$ , since the ring of polynomials with real coefficients  $\mathbb{R}[s]$  is isomorphic to  $C_{p\text{-imp}}$  [25, Proposition 2.3]. For instance, the polynomial  $p(s) = 2 - 3s + s^2$  corresponds to the pulse  $p(\delta^{(1)}) = 2\delta - 3\delta^{(1)} + \delta^{(2)}$  (recall that  $\delta^{(0)} = \delta$ ,  $\delta^{(2)} = \delta^{(1)} * \delta^{(1)}$ , etc.). The rational function  $r(s) = s/(s - 2)$  corresponds to the fractional pulse  $r(\delta^{(1)}) = \delta^{(1)} * (\delta^{(1)} - 2\delta)^{-1} = \delta^{(1)} * u = \dot{u} + u(0^+)\delta$  with  $u = \exp(2t)$  on  $\mathbb{R}^+$ . Consequently, if  $k_1, k_2$  are any two nonnegative integers,  $M^{k_1 \times k_2}(s)$ ,  $M_f^{k_1 \times k_2}(\delta^{(1)})$  denote the sets of  $k_1 \times k_2$  matrices with entries in  $\mathbb{R}(s)$ ,  $C_f$ , respectively, and  $T(s) \in M^{k_1 \times k_2}(s)$ ,  $T(\delta^{(1)})$  is the corresponding element in  $M_f^{k_1 \times k_2}(\delta^{(1)})$ , then  $T(s)$  is left (right) invertible as a rational matrix if and only if  $T(\delta^{(1)})$  is left (right) invertible as a matrix with entries in  $C_f$  [25, Corollary 2.4]. Also, note that  $C_{\text{imp}}$  is a commutative ring.

Remark 3.5.

Apart from the claim that  $\mathcal{W}(\Sigma)$  is the smallest subspace  $\mathcal{L}$  that satisfies (3.4), it is proven in [25] (Corollary 3.13) that  $\mathcal{W}(\Sigma)$  is the smallest subspace  $\mathcal{L}$  for which there exists a  $G \in \mathbb{R}^{l \times r}$  such that

$$E^{-1}\{(A + GC)\mathcal{L} + \text{im}(B + GD)\} \subset \mathcal{L}.$$

A Molinari-type algorithm for computing  $\mathcal{W}(\Sigma)$ , following directly from [25, Theorem 3.9], is given in [25, Theorem 3.10]. Unlike in [20], we allow  $E$  and  $A$  to be nonsquare. If  $D = 0$ ,  $\mathcal{W}(\Sigma)$  may be called the infimal  $(C, A, E)$ -invariant subspace related to  $\text{im}(B)$ . If  $E = I$ , then [25, Theorem 3.9, Theorem 3.10, Corollary 3.13] reduce to [8, (3.14), (3.22), Theorem 3.15], respectively.

Similar subspace conditions and algorithms for the discrete-time case are presented in [27]. Note that the limiting subspace of the sequence  $\{x_k\}$  in [27] equals our  $\mathcal{W}(\Sigma)$ , as is to be expected [8].

Remark 3.6.

In [12, Section 3, Assumption 2] it is assumed that (in terms of (3.5))  $[C_2 \ D]$  is left invertible. Hence the problems considered there are indeed regular in the sense of our Definition 3.1. However, it is very well possible that the LQCP defined in [12, Sections 1, 2] is regular even if  $[C_2 \ D]$  is not of full column rank. For instance, consider the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \delta^{(1)} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \delta,$$

$$y = [1 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Clearly,  $[C_2 \ D] = [1 \ 0]$  is not left invertible, but  $\begin{bmatrix} A_{22} & B_2 \\ C_2 & D \end{bmatrix}$  is.

Hence the LQCP associated with this system is regular in the sense of Definition 3.1. Indeed, the control  $u = x_2$  yields  $\delta^{(1)} * x_1 = -x_1 + x_{01}\delta$  and hence  $x_1 = (\delta^{(1)} + \delta)^{-1} * x_{01}\delta$ , i.e.,  $x_1(t) = \exp(-t)x_{01}$  on  $\mathbb{R}^+$ ,  $x_2 = -x_1$  and  $y = 0$ . We establish that in [12] only a special class of regular nonnegative definite LQCPs subject to implicit systems has been solved; we will solve the general case (i.e., without any unnecessary assumptions such as [12, (56)], left-invertibility of  $[C_2 \ D]$ ) in a future paper.

Remark 3.7.

In [25] several invertibility concepts for general implicit systems have been defined and analyzed. There, a system  $\Sigma$  (2.5a) - (2.5b) is called *left invertible in the strong sense* if

$$x_0 = 0, y = 0 \Rightarrow Ex = 0, u = 0$$

(and left invertible in the *weak* sense if  $x_0 = 0, y = 0 \Rightarrow u = 0$ ) [25, Section 4]. Assume that  $[E' \ A' \ C']'$  is of full column rank.

Then the following statements are equivalent [25, Corollary 4.15].

- i)  $\Sigma$  is left invertible in the strong sense.
- ii) If  $x_0 = 0$ ,  $y = 0$ , then  $x = 0$ ,  $u = 0$ .
- iii)  $P_\Sigma(s)$  is left invertible as a rational matrix.

In the proof of Theorem 3.2 we saw that the condition (3.3) is sufficient for left-invertibility of  $P_\Sigma(s)$  and hence we observe that  $\Sigma$  is left invertible in the strong sense if (3.3) is satisfied. It follows that a certain LQCP is regular only if the underlying system is left invertible (in the strong sense), by Theorem 3.2. The converse is not true, of course [8]. Note that left-invertibility of  $P_\Sigma(s)$  is equivalent to left-invertibility of the transfer function  $T(s) := D + C(sE - A)^{-1}B$  if  $\det(sE - A) \neq 0$  [24, Theorem 3.9], [8, Theorem 3.26].

Corollary 3.8.

Assume that  $\ker \begin{pmatrix} E & 0 \\ A & B \\ C & D \end{pmatrix} = 0$ . Then

$$\mathfrak{W}(\Sigma) = \ker(E) \iff \ker \begin{pmatrix} E & 0 \\ C & D \end{pmatrix} \cap [A \ B]^{-1} \text{im}(E) = 0.$$

Proof.  $\Leftarrow$  Follows from (3.4).  $\Rightarrow$  If  $E\bar{x} = 0$ ,  $C\bar{x} + D\bar{u} = 0$  and  $A\bar{x} + B\bar{u} = Ew$ , then  $(-w) \in \mathfrak{W}(\Sigma)$  since  $x = \bar{x}\delta \in S(-w, \bar{u}\delta)$ . Hence  $A\bar{x} + B\bar{u} = 0$  and  $\bar{x} = 0$ ,  $\bar{u} = 0$ .

The assumption in Corollary 3.8 is *not* necessarily satisfied if  $\ker(E) = \mathfrak{W}(\Sigma)$  for an arbitrary system  $\Sigma$ . Take e.g.  $E = I$ ,  $B = 0$ ,  $C = I$  and  $D = 0$ , then, obviously,  $\mathfrak{W}(\Sigma) = \ker(E)$ , but  $\begin{bmatrix} B \\ D \end{bmatrix}$  is not left invertible. However, without loss of

generality, one can assume that  $\ker \begin{pmatrix} E & 0 \\ A & B \\ C & D \end{pmatrix} = 0$  in (2.5a) -

(2.5b) and if this is the case, then an alternative characterization of regularity might be: The LQCP (2.5) - (2.6) is regular if and only if  $\mathfrak{W}(\Sigma) = \ker(E)$ .



### Conclusions.

Our distributional framework covers all existing interpretations of continuous-time linear-quadratic control problems subject to general systems. We saw that within this distributional context the concept of regularity can be understood in a very natural way as the property that the output is a function only if inputs and state trajectories are, not only in the standard but also in the nonstandard cases. We derived a condition that is equivalent to this property and since this condition is expressed in the (unrestricted) system coefficients only, it is easily checked. Moreover, we related this condition to the strongly controllable subspace and established that, without loss of generality, LQCPs are regular if and only if this subspace is trivial. Finally, we noted that in the existing literature only special cases of regular LQCPs subject to implicit systems have been treated. The author wants to discuss problems subject to arbitrary systems in a future article.

### Illustrative Examples.

Consider the system equation

$$[1 \ 0] \delta^{(1)} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1 \ 0] \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \delta,$$

with output  $y_1 = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$ . Then the condition (3.3) is not satisfied; if e.g.  $u = \delta$ ,  $x_2 = -\delta$ , then  $y_1 = 0$ , smooth. If  $y_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ , then (3.3) holds. Indeed,  $y_2$  is a function only if  $u$  and  $x_2$  are, as a result of which  $x_1$  is a function as well. If  $y_3 = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  ( $B$  and  $D$  are not appearing), then (3.3) is valid and, again,  $y_3$  is a function only if inputs and states are.

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